b) pour $x \in]0;1]$

$g(x) = \frac{\sqrt{1+x^2}-1}{x} = \frac{(1+x^2)-1}{x(\sqrt{1+x^2}+1)} = \frac{x}{1+\sqrt{1+x^2}}$

$f(x) \le \frac{1}{2}x$ d'aprés 1) c)

$\Rightarrow g(x) = \frac{1}{2}x$ d'où $g(U_n) \le \frac{1}{2}U_n$

$\Rightarrow U_{n+1} \le \frac{1}{2}U_n \quad \forall n \in \mathbb{N}$

c) * pour n=0 $U_0 = 1 \le \frac{1}{2^0} = 1$ (vrai)

* supposons que $U_n \le (\frac{1}{2})^n$

* montrons que : $U_{n+1} \le (\frac{1}{2})^{n+1}$

n a : $U_{n+1} \le \frac{1}{2}U_n$ et $U_n \le (\frac{1}{2})^n \Rightarrow U_{n+1} \le (\frac{1}{2})^{n+1}$

onclusion : $U_n \le \frac{1}{2^n}$; $\forall n \in \mathbb{N}$

$0 \le U_n \le (\frac{1}{2})^n$

$\lim_{+\infty}(\frac{1}{2})^n = 0$ d'où $\lim_{x\to+\infty} U_n = 0$

) a) $U_{n+1} = f(U_n) \Rightarrow U_n = f^{-1}(U_{n+1})$

$= \frac{2U_{n+1}}{1-(U_{n+1})^2}$

b) * pour n=0 , $U_0 = 1 = tg(\frac{\pi}{2^{0+2}})$ (vrai)

* supposons que : $U_n = tg(\frac{\pi}{2^{n+2}})$

* montrons que : $U_{n+1} = tg(\frac{\pi}{2^{n+3}})$

$U_{n+1} = g(U_n) = \frac{U_n}{1+\sqrt{1+U_n^2}}$

$= \frac{U_n}{1+\sqrt{1+tg^2(\frac{\pi}{2^{n+2}})}}$ $\quad (1+\text{tg}^2 x = \frac{1}{\cos^2 x})$

$U_{n+1} = \frac{U_n}{1+\frac{1}{\cos^2(\frac{\pi}{2^{n+2}})}} = \frac{\cos(\frac{\pi}{2^{n+2}})tg(\frac{\pi}{2^{n+2}})}{1+\cos(\frac{\pi}{2^{n+2}})}$

$= \frac{\sin(\frac{\pi}{2^{n+2}})}{2\cos^2(\frac{\pi}{2^{n+2}})} = \frac{2\sin(\frac{\pi}{2^{n+3}})\cos(\frac{\pi}{2^{n+3}}).}{2\cos^2(\frac{\pi}{2^{n+3}})}$

car : $1+\cos x = 2\cos^2\frac{x}{2}$; $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

d'où $U_n = tg(\frac{\pi}{2^{n+3}})$

conclusion : $U_n = tg(\frac{\pi}{2^{n+2}})$; $\forall n \in \mathbb{N}$

3) $V_n = 2^n.U_n = 2.tg(\frac{\pi}{2^{n+2}}) = 2^n tg(\frac{1}{2^n}.\frac{\pi}{4})$

$V_n = \frac{tg(\frac{1}{2^n}.\frac{\pi}{4})}{1/2^n} = h(\frac{1}{2^n})$ avec $h(x) = \frac{\text{tg}(\frac{\pi}{4}x)}{x}$

$\lim_{n\to+\infty}\frac{1}{2^n} = 0$

$\lim_{n\to 0}\frac{tg(\frac{\pi}{4}x)}{x} = \frac{\pi}{4}$

d'où $\lim_{x\to+\infty} V_n = \frac{\pi}{4}$

Qarray.tn -Aqra Blech- Qarray.tn